Konuralp Journal of Mathematics
Volume 5 No. 1 pp. 176–186 (2017) ©KJM

ON THE FEKETE-SZEGÖ PROBLEM FOR ANALYTIC FUNCTIONS DEFINED BY USING SYMMETRIC Q-DERIVATIVE OPERATOR

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Abstract. The aim of this paper is to establish the Fekete-Szegö inequalities for two new subclasses of analytic functions which are associated with symmetric q-derivative operator.

1. Notations and Definitions in q-Calculus

First formulae in what we now call q-calculus were obtained by Euler in the eighteenth century. In the second half of the twentieth century there was a significant increase of activity in the area of the q-calculus. The fractional calculus operators has gained importance and popularity, mainly due to its vast potential of demonstrated applications in various fields of applied sciences, engineering. The application of q-calculus was initiated by Jackson [9].

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$. We shall follow the notation and terminology in [8] and [14]. We recall the definitions of fractional q-calculus operators of a complex-valued function $f(z)$.

Definition 1.1. Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^n}{1-q} & (\lambda \in \mathbb{C}) \\ \sum\limits_{n=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Date: June 1, 2016 and, in revised form, January 25, 2017.
2000 *Mathematics Subject Classification.* 30C45, 05A30, 33D15.
Key words and phrases. Analytic functions, bi-univalent functions, Fekete-Szegö functional, q-derivative operator.

Definition 1.2. Let $q \in (0,1)$ and define the q-fractional $[n]_q!$ by

$$[n]_q! = \begin{cases} \prod_{k=1}^{n} [k]_q, & (n \in \mathbb{N}) \\ 1, & (n = 0). \end{cases}$$

for $n \in \mathbb{N}$.

Definition 1.3. For $q \in (0,1)$, $\lambda, \eta \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the q-shifted factorial $(\lambda; q)_\eta$ is defined by

$$(\lambda; q)_\eta = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\eta + j}} \right)$$

so that

$$(\lambda; q)_n = \begin{cases} \prod_{j=0}^{n-1} \left(1 - \lambda q^j\right) & (n \in \mathbb{N}) \\ 1, & (n = 0). \end{cases}$$

and

$$(\lambda; q)_\infty = \prod_{j=0}^{\infty} (1 - \lambda q^j).$$

Definition 1.4. (see [9]; see also [8], [14]) The q-derivative of a function f is defined in a given subset of $\mathbb{C}$ is by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad if \quad z \neq 0, \tag{1.1}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = \frac{df(z)}{dz}$$

if f is differentiable.

The aim of this paper is to establish the Fekete-Szegö inequality for two new subclasses of univalent and bi-univalent functions which are associated with symmetric q-derivative operator.

2. Fekete-Szegö problem for a new subclass of univalent functions

Let A represent the class of analytic functions in the unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

From (1.1), we deduce that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}.$$

The Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ for normalized univalent functions given by (2.1) is well known for its rich history in Geometric Function Theory. For $f \in S$ and given by (2.1), that

$$\left|a_3 - a_2^2\right| \leq 1$$

where the equality holds true for the Koebe function:

$$k(z) = \frac{z}{(1-z)^2}.$$

Earlier in 1933, Fekete and Szegö [7] made use of Lowner's parametric method in order to prove that, if $f \in S$ and is given by (2.1),

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) \quad (0 \leq \mu \leq 1).$$

The functional has since received great attention, particularly in many subclasses of the families of univalent and bi-univalent functions. Nowadays, it seems that this topic had become an interest among the researchers (see, for example, [1], [2], [5], [10], [13]).

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as

$$f(z) \prec g(z), \qquad (z \in U)$$

if there exists a Schwarz function $w(z)$, analytic in U, with

$$w(0) = 0 \ \text{ and } \quad |w(z)| < 1 \qquad (z \in U)$$

such that

$$f(z) = g(w(z)) \qquad (z \in U).$$

Let P denote the class of functions consisting of p, such that

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are regular in the open unit disc U and satisfy $\Re(p(z)) > 0$ for any $z \in U$. Here, $p(z)$ is called Caratheodory function [6].

Let ϕ be an analytic and univalent function with positive real part in U with $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps the unit disc U onto a region starlike with respect to 1, and symmetric with respect to the real axis. The Taylor's series expansion of such function is of the form

$$\phi(z) = 1 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots. \tag{2.2}$$

where all coefficients are real and $C_1 > 0$.

Definition 2.1. (see [3]) The symmetric q-derivative $\widetilde{D}_q f$ of a function f given by (2.1) is defined as follows:

$$\left(\widetilde{D}_q f\right)(z) = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z}, \quad if \quad z \neq 0, \tag{2.3}$$

and $(\widetilde{D}_q f)(0) = f'(0)$ provided $f'(0)$ exists.

From (2.3), we deduce that

$$\left(\widetilde{D}_q f\right)(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1}, \tag{2.4}$$

where the symbol $\widetilde{[n]}_q$ denotes the number

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

frequently occurring in the study of q-deformed quantum mechanical simple harmonic oscillator (see [4]).

The following properties hold

$$\widetilde{D}_q(f(z) + g(z)) = \left(\widetilde{D}_q f\right)(z) + \left(\widetilde{D}_q g\right)(z)$$

$$\begin{aligned} \widetilde{D}_q(f(z)g(z)) &= g(q^{-1}z)\left(\widetilde{D}_q f\right)(z) + f(qz)\left(\widetilde{D}_q g\right)(z) \\ &= g(qz)\left(\widetilde{D}_q f\right)(z) + f(q^{-1}z)\left(\widetilde{D}_q g\right)(z) \end{aligned}$$

$$\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}.$$

Finally, we have the following relation

$$\left(\widetilde{D}_q f\right)(z) = \left(D_{q^2} f\right)(q^{-1}z).$$

Definition 2.2. A function $f \in A$ is said to be in the class $N(q, \phi)$ if it satisfies the following subordination condition:

$$\frac{z\left(\widetilde{D}_q f\right)(z)}{f(z)} \prec \phi(z) \quad (z \in U)$$

where the operator $\widetilde{D}_q f$ is given by (2.3).

We note that

$$\lim_{q \to 1^-} N(q; \phi) = \left\{ f \in A : \lim_{q \to 1^-} \frac{z\left(\widetilde{D}_q f\right)(z)}{f(z)} \prec \phi(z), \quad z \in U \right\} = S^*(\phi)$$

where $S^*(\phi)$ is the class of Ma-Minda starlike functions defined by Ma and Minda [11].

In order to derive our main result, we require the following lemmas.

Lemma 2.1. *(see [12]) If $p \in P$, then*

$$|p_n| \le 2, \qquad n \in \mathbb{N}$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \le 2 - \frac{|p_1|^2}{2}.$$

Lemma 2.2. *(see [11]) If $p \in P$, then*

$$\left| p_2 - t p_1^2 \right| \le \begin{cases} -4t + 2; & if \quad t \le 0 \\ 2; & if \quad 0 \le t \le 1 \\ 4t - 2; & if \quad t \ge 1 \end{cases}.$$

When $t<0$ or $t>1$, the equality holds if and only if

$$p(z)=\frac{1+z}{1-z},$$

or one of its rotations. When $0<t<1$, then the equality holds if and only if

$$p(z)=\frac{1+z^2}{1-z^2},$$

or one of its rotations. If $t=0$, the equality holds if and only if

$$p(z)=\left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z}\qquad(0\leq\lambda\leq 1),$$

or one of its rotations. If $t=1$, the equality holds if and only if p is the reciprocal of one of the functions such that equality holds in the case of $t=0$. Also the above upper bound is sharp, and it can be improved as follows when $0<t<1$:

$$\left|p_2-tp_1^2\right|+t\left|p_1\right|^2\leq 2\qquad\left(0<t\leq\frac{1}{2}\right),$$

$$\left|p_2-tp_1^2\right|+(1-t)\left|p_1\right|^2\leq 2\qquad\left(\frac{1}{2}<t\leq 1\right).$$

By using Lemma 2.2, we have the following theorem:

Theorem 2.1. *Let f given by (2.4) be in the class $N(q,\phi)$. Then*
(2.5)

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{C_2}{\widetilde{[3]}_q-1}-\frac{\mu C_1^2}{\left(\widetilde{[2]}_q-1\right)^2}+\frac{C_1^2}{\left(\widetilde{[2]}_q-1\right)\left(\widetilde{[3]}_q-1\right)};\\ \quad for\quad \mu\leq\frac{\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[3]}_q-1}\left(\frac{C_2-C_1}{C_1^2}\right)+\frac{\widetilde{[2]}_q-1}{\widetilde{[3]}_q-1}\\ \frac{C_1}{\left(\widetilde{[3]}_q-1\right)};\\ \quad for\ \frac{\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[3]}_q-1}\left(\frac{C_2-C_1}{C_1^2}\right)+\frac{\widetilde{[2]}_q-1}{\widetilde{[3]}_q-1}\leq\ \mu\leq\frac{\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[3]}_q-1}\left(\frac{C_2+C_1}{C_1^2}\right)+\frac{\widetilde{[2]}_q-1}{\widetilde{[3]}_q-1}\\ -\frac{C_2}{\widetilde{[3]}_q-1}+\frac{\mu C_1^2}{\left(\widetilde{[2]}_q-1\right)^2}-\frac{C_1^2}{\left(\widetilde{[2]}_q-1\right)\left(\widetilde{[3]}_q-1\right)};\\ \quad for\quad \mu\geq\frac{\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[3]}_q-1}\left(\frac{C_2+C_1}{C_1^2}\right)+\frac{\widetilde{[2]}_q-1}{\widetilde{[3]}_q-1}\end{cases}.$$

The result is sharp.

Proof. Let $f\in N(q,\phi)$. Then there exist a function u, analytic in U with $u(0)=0,\ |u(z)|<1,\ z\in U$ such that

$$\frac{z\left(\widetilde{D}_qf\right)(z)}{f(z)}=\phi\left(u\left(z\right)\right),\quad z\in U. \tag{2.6}$$

Next, define the function p by

$$p\left(z\right)=\frac{1+u\left(z\right)}{1-u\left(z\right)}=1+p_1z+p_2z^2+\cdots. \tag{2.7}$$

Clearly, $\Re(p(z)) > 0$. From (2.7), one can derive

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right)z^2 + \cdots . \tag{2.8}$$

Combining (2.2), (2.6) and (2.8),

$$\frac{z\left(\widetilde{D}_q f\right)(z)}{f(z)} = 1 + \frac{1}{2}C_1 p_1 z + \left(\frac{1}{4}C_2 p_1^2 + \frac{1}{2}C_1\left(p_2 - \frac{1}{2}p_1^2\right)\right)z^2 + \cdots \tag{2.9}$$

From (2.9), we deduce

$$\left(\widetilde{[2]}_q - 1\right)a_2 = \frac{1}{2}C_1 p_1, \tag{2.10}$$

$$\left(\widetilde{[3]}_q - 1\right)a_3 - \left(\widetilde{[2]}_q - 1\right)a_2^2 = \frac{1}{4}C_2 p_1^2 + \frac{1}{2}C_1\left(p_2 - \frac{1}{2}p_1^2\right). \tag{2.11}$$

From (2.10) and (2.11) it follows that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{C_1}{2\left(\widetilde{[3]}_q - 1\right)}\left\{p_2 - \frac{p_1^2}{2}\left[1 - \frac{C_2}{C_1} + \frac{\mu C_1}{\left(\widetilde{[2]}_q - 1\right)^2}\left(\widetilde{[3]}_q - 1\right) - \frac{C_1}{\widetilde{[2]}_q - 1}\right]\right\} \\ &= \frac{C_1}{2\left(\widetilde{[3]}_q - 1\right)}\left(p_2 - t p_1^2\right), \end{aligned}$$

where

$$t = \frac{1}{2}\left[1 - \frac{C_2}{C_1} + \frac{\mu\left(\widetilde{[3]}_q - 1\right) - \left(\widetilde{[2]}_q - 1\right)}{\left(\widetilde{[2]}_q - 1\right)^2}C_1\right].$$

Then, applying Lemma 2.2, the proof is completed. $\square$

3. Fekete-Szegö problem for a new subclass of bi-univalent functions

The Koebe one-quarter theorem [6] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots . \tag{3.1}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ, see [15].

From (2.3) and (3.1), we also deduce that

$$
\begin{aligned}
(\widetilde{D}_q g)(w) &= \frac{g(qw)-g(q^{-1}w)}{(q-q^{-1})w} \\
&= 1\ - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q \left(2a_2^2 - a_3\right) w^2 \qquad (3.2)\\
&\quad - \widetilde{[4]}_q \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^3 + \cdots .
\end{aligned}
$$

Definition 3.1. A function $f \in \Sigma$ is said to be in the class $N_\Sigma(q;\phi)$, if the following subordinations hold

$$\frac{z\left(\widetilde{D}_q f\right)(z)}{f(z)} \prec \phi(z), \qquad (\ z \in U)$$

and

$$\frac{w\left(\widetilde{D}_q g\right)(w)}{g(w)} \prec \phi(w), \qquad (w \in U)\,.$$

where $g = f^{-1}$.

We note that

$$\lim_{q\to 1^-} N_\Sigma(q;\phi) = \left\{ f \in \Sigma : \begin{array}{c} \lim_{q\to1^-} \frac{z\left(\widetilde{D}_q f\right)(z)}{f(z)} \prec \phi(z), \quad z \in U \\ \\ \lim_{q\to1^-} \frac{w\left(\widetilde{D}_q g\right)(w)}{g(w)} \prec \phi(w), \quad w \in U \end{array} \right\} = S^*_\Sigma(\phi)$$

where $S^*_\Sigma(\phi)$ is the class of Ma-Minda bi-starlike functions defined by Ma and Minda [11].

Theorem 3.1. *Let f given by (2.1) be in the class $N_\Sigma(q;\phi)$ and $\mu \in \mathbb{R}$. Then*

$$
\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{C_1}{\widetilde{[3]}_q - 1}; \\ \qquad for \quad |\mu-1| \le \frac{1}{\widetilde{[3]}_q-1}\left|\widetilde{[3]}_q - \widetilde{[2]}_q + \left(\widetilde{[2]}_q - 1\right)^2 \frac{(C_1-C_2)}{C_1^2}\right| \\ \\ \frac{C_1^3|\mu-1|}{\left|\left(\widetilde{[3]}_q - \widetilde{[2]}_q\right)C_1^2 + \left(\widetilde{[2]}_q-1\right)^2 (C_1-C_2)\right|}; \\ \\ \qquad for \quad |\mu-1| \ge \frac{1}{\widetilde{[3]}_q-1}\left|\widetilde{[3]}_q - \widetilde{[2]}_q + \left(\widetilde{[2]}_q - 1\right)^2 \frac{(C_1-C_2)}{C_1^2}\right|. \end{cases} \tag{3.3}
$$

Proof. Let $f \in N_\Sigma(q;\phi)$ and g be the analytic extension of f^{-1} to U. Then there exist two functions u and v, analytic in U with $u(0) = v(0) = 0,\ |u(z)| < 1,\ |v(w)| < 1,\ z, w \in U$ such that

$$\frac{z\left(\widetilde{D}_q f\right)(z)}{f(z)} = \phi\left(u\left(z\right)\right), \quad (z \in U) \tag{3.4}$$

and

$$\frac{w\left(\widetilde{D}_q g\right)(w)}{g(w)} = \phi\left(v\left(w\right)\right), \quad (w \in U)\,. \tag{3.5}$$

Next, define the functions p and t by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots \tag{3.6}$$

and

$$t(w)=\frac{1+v(w)}{1-v(w)}=1+t_1w+t_2w^2+\cdots. \tag{3.7}$$

Clearly, $\Re(p(z))>0$ and $\Re(t(w))>0$. From (3.6), (3.7) one can derive

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2\right)z^2+\cdots \tag{3.8}$$

and

$$v(w)=\frac{t(w)-1}{t(w)+1}=\frac{1}{2}t_1w+\frac{1}{2}\left(t_2-\frac{1}{2}t_1^2\right)w^2+\cdots. \tag{3.9}$$

Combining (2.2), (3.6), (3.7), (3.8) and (3.9),

$$\frac{z\left(\widetilde{D}_qf\right)(z)}{f(z)}=1+\frac{1}{2}C_1p_1z+\left(\frac{1}{4}C_2p_1^2+\frac{1}{2}C_1\left(p_2-\frac{1}{2}p_1^2\right)\right)z^2+\cdots \tag{3.10}$$

and

$$\frac{w\left(\widetilde{D}_qg\right)(w)}{g(w)}=1+\frac{1}{2}C_1t_1w+\left(\frac{1}{4}C_2t_1^2+\frac{1}{2}C_1\left(t_2-\frac{1}{2}t_1^2\right)\right)w^2+\cdots. \tag{3.11}$$

From (3.10) and (3.11), we deduce

$$\left(\widetilde{[2]}_q-1\right)a_2=\frac{1}{2}C_1p_1, \tag{3.12}$$

$$\left(\widetilde{[3]}_q-1\right)a_3-\left(\widetilde{[2]}_q-1\right)a_2^2=\frac{1}{4}C_2p_1^2+\frac{1}{2}C_1\left(p_2-\frac{1}{2}p_1^2\right). \tag{3.13}$$

and

$$-\left(\widetilde{[2]}_q-1\right)a_2=\frac{1}{2}C_1t_1, \tag{3.14}$$

$$\left(\widetilde{[3]}_q-1\right)\left(2a_2^2-a_3\right)-\left(\widetilde{[2]}_q-1\right)a_2^2=\frac{1}{4}C_2t_1^2+\frac{1}{2}C_1\left(t_2-\frac{1}{2}t_1^2\right). \tag{3.15}$$

From (3.12) and (3.14) we obtain

$$p_1=-t_1. \tag{3.16}$$

Subtracting (3.13) from (3.15) and applying (3.16) we have

$$a_3=a_2^2+\frac{1}{4\left(\widetilde{[3]}_q-1\right)}C_1\left(p_2-t_2\right). \tag{3.17}$$

By adding (3.13) to (3.15), we get

$$a_2^2=\frac{C_1^3\left(p_2+t_2\right)}{4\left[\left(\widetilde{[3]}_q-\widetilde{[2]}_q\right)C_1^2+\left(\widetilde{[2]}_q-1\right)^2\left(C_1-C_2\right)\right]}. \tag{3.18}$$

From (3.17) and (3.18) it follows that

$$a_3-\mu a_2^2=C_1\left[\left(h(\mu)+\frac{1}{4\left(\widetilde{[3]}_q-1\right)}\right)p_2+\left(h(\mu)-\frac{1}{4\left(\widetilde{[3]}_q-1\right)}\right)t_2\right],$$

where

$$h(\mu)=\frac{C_1^2(1-\mu)}{4\left[\left(\widetilde{[3]}_q-\widetilde{[2]}_q\right)C_1^2+\left(\widetilde{[2]}_q-1\right)^2(C_1-C_2)\right]}.$$

Then, applying Lemma 2.1, we conclude that

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{C_1}{\widetilde{[3]}_q-1} & for\quad 0\leq|h(\mu)|\leq\frac{1}{4\left(\widetilde{[3]}_q-1\right)}\\ 4C_1|h(\mu)| & for\quad |h(\mu)|\geq\frac{1}{4\left(\widetilde{[3]}_q-1\right)}.\end{cases}$$

Taking $\mu=1$ or $\mu=0$ we get

Corollary 3.1. *If* $f\in\mathrm{N}_\Sigma(q,\phi)$ *then*

$$\left|a_3-a_2^2\right|\leq\frac{C_1}{\widetilde{[3]}_q-1}. \tag{3.19}$$

Corollary 3.2. *If* $f\in\mathrm{N}_\Sigma(q,\phi)$ *then*

$$|a_3|\leq\begin{cases}\frac{C_1}{\widetilde{[3]}_q-1}; & for\quad \frac{C_1-C_2}{C_1^2}\in\left(-\infty,\frac{\widetilde{[2]}_q-2\widetilde{[3]}_q+1}{\left(\widetilde{[2]}_q-1\right)^2}\right]\cup\left[\frac{1}{\widetilde{[2]}_q-1},\infty\right)\\ \frac{C_1^3}{\left|\left(\widetilde{[3]}_q-\widetilde{[2]}_q\right)C_1^2+\left(\widetilde{[2]}_q-1\right)^2(C_1-C_2)\right|}; & \\ for\quad \frac{C_1-C_2}{C_1^2}\in\left[\frac{\widetilde{[2]}_q-2\widetilde{[3]}_q+1}{\left(\widetilde{[2]}_q-1\right)^2},\frac{\widetilde{[2]}_q-\widetilde{[3]}_q}{\left(\widetilde{[2]}_q-1\right)^2}\right)\cup\left(\frac{\widetilde{[2]}_q-\widetilde{[3]}_q}{\left(\widetilde{[2]}_q-1\right)^2},\frac{1}{\widetilde{[2]}_q-1}\right]. & \end{cases} \tag{3.20}$$

Corollary 3.3. *If we let*

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^\alpha=1+2\alpha z+2\alpha^2z^2+\ldots\qquad(0<\alpha\leq1),$$

then inequalities (3.19) and (3.20) become

$$\left|a_3-a_2^2\right|\leq\frac{2\alpha}{\widetilde{[3]}_q-1},$$

and

$$|a_3|\leq\begin{cases}\frac{2\alpha}{\widetilde{[3]}_q-1}; & \alpha\in\left(0,\frac{\widetilde{[2]}_q-1}{\widetilde{[2]}_q+1}\right]\cup\left[\frac{\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[2]}_q^2-4\widetilde{[3]}_q+3},1\right]\\ \frac{4\alpha^2}{\left(2\widetilde{[3]}_q-\widetilde{[2]}_q^2-1\right)\alpha+\left(\widetilde{[2]}_q-1\right)^2}; & \alpha\in\left[\frac{\widetilde{[2]}_q-1}{\widetilde{[2]}_q+1},\frac{\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[2]}_q^2-2\widetilde{[3]}_q+1}\right)\cup\left(\frac{\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[2]}_q^2-2\widetilde{[3]}_q+1},\frac{\left(\widetilde{[2]}_q-1\right)^2}{\widetilde{[2]}_q^2-4\widetilde{[3]}_q+3}\right].\end{cases}$$

Corollary 3.4. *If we let*

$$\phi(z)=\frac{1+(1-2\alpha)z}{1-z}=1+2(1-\alpha)z+2(1-\alpha)z^2+\cdots\qquad(0\leq\alpha<1),$$

then inequalities (3.19) and (3.20) become

$$\left|a_3 - a_2^2\right| \leq \frac{2\,(1-\alpha)}{\widetilde{[3]}_q - 1},$$

and

$$|a_3| \leq \frac{2\,(1-\alpha)}{\widetilde{[3]}_q - \widetilde{[2]}_q}.$$

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