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Research Article

Quasi subordinations for bi-univalent functions with symmetric conjugate points

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ABSTRACT

Many researchers have recently acquainted and researched several interesting subfamilies of bi-univalent function family δ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. In this current work, the subfamily $\mathcal{F}_{\delta,q}^{sc}(\alpha,\Xi)$ of bi-univalent functions in the sense of symmetric conjugate points with quasi subordination is defined. The Maclaurin coefficients $|a_2|$, $|a_3|$ and besides related with these coefficients $|a_3-a_2{}^2|$ for functions belonging to this subfamily are derived. Further some corollaries are also presented.

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INTRODUCTION

Let $\mathcal{A}$ be the family of functions s of the type

$$s(z)=z+a_2z^2+a_2z^3+\cdots, \tag{1}$$

which is analytic in the open disc $\Omega=\{z\in\mathbb{C}:|z|<1\}$, and holds normalized equalities $s(0)=s'(0)-1=0$.

Furthermore, we indicate all analytic function family which are univalent in Ω by S. Let's take $t(z)$ be an analytic function in Ω, such thatt

$$t(z)=t_0+t_1z+t_2z^2+\cdots,\quad |t(z)|\le 1, \tag{2}$$

where all coefficients are real. Besides, let the function Ξ be an analytic and univalent function with positive real part in Ω with $\Xi\,(0)=1$, $\Xi'(0)>0$ and Ξ maps the unit disc Ω onto a region starlike in the sense of $\Xi\,(0)=1$ and symmetric in the sense of real axis. Taylor's series expansion for such a function is of the type

$$\Xi\,(z)=1+A_1z+A_2z^2+\cdots, \tag{3}$$

where all coefficients are real and $A_1>0$.

For two functions s and r analytic in Ω, we say that the function s is subordinate to r, written $s\prec r$ or $s(z)\prec r(z)$ $(z\in\Omega)$, if there exist a Schwarz function w analytic in Ω with $w(0)=0$ and $|w(z)|<1$, such that $s(z)=r(w(z))$, $z\in\Omega$.

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Robertson [12] acquainted the notion of quasi-subordination in 1970. For two analytic functions s and r, the function s is said to be quasi-subordinate to r in Ω and written as

$$s(z) \prec_q r(z) \quad (z \in \Omega),$$

if there exists an analytic function $|t(z)| \leq 1, z \in \Omega$ such that $\frac{s(z)}{t(z)}$ analytic in Ω,

$$\frac{s(z)}{t(z)} \prec r(z), \ \ z \in \Omega.$$

Namely, there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| \leq |z|$ such that

$$s(z) = t(z) r\big(w(z)\big), \ \ z \in \Omega.$$

Observe that if $t(z) = 1$, then $s(z) = r(w(z))$, so that $s(z) \prec r(z)$ in Ω. Besides, realize that if $w(z) = z$, then $s(z) = t(z)\, r(z)$, and it is said that s is majorized by r and written $s(z) \ll r(z)$ in Ω. Therefore it is apparent that quasi-subordination is a generalization of subordination as well as majorization (see e.g. [1, 3, 6, 10-12] for works concerned with quasi-subordination and subordination).

The Koebe-one quarter theorem [5] guarantees that the image of Ω under every univalent function $s \in \mathcal{A}$ comprises a disc of radius $\frac{1}{4}$. Hence every univalent function s has an inverse s^{-1} providing $s^{-1}(s(z)) = z, z \in \Omega$ and $s(s^{-1}(\mathrm{w})) = \mathrm{w}$, $\left(|w| < r_0(s),\ r_0(s) \geq \frac{1}{4}\right)$. Indeed, the inverse function s^{-1} is given by

$$s^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \quad (4)$$

If both s and s^{-1} are univalent in Ω, then the function $s \in \mathcal{A}$ is said to be bi-univalent in Ω. Let δ indicate the family of bi-univalent functions defined in Ω.

Many researchers have recently acquainted and researched several interesting subfamilies of bi-univalent function family δ, and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ [2, 4, 13-15, 17]. However, we have a few study about the general coefficient bounds $|a_n|$ for bi- univalent functions in the literature [8, 9]. There is still an obvious problem for $|a_n|$, $n \in \mathbb{N} - \{1, 2, 3\}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$. El-Ashwah and Thomas [7] acquainted the family S^* of functions named starlike in the sense of symmetric conjugate points, they are the functions $s \in S$ provide the inequality

$$Re\left\{\frac{zs'(z)}{s(z) - \overline{s(-\overline{z})}}\right\} > 0, \ \ z \in \Omega.$$

If

$$Re\left\{\frac{(zs'(z))'}{(s(z) - \overline{s(-\overline{z})})'}\right\} > 0, \quad z \in \Omega$$

then, the function $s \in S$ is named convex in the sense of symmetric conjugate points. The family of all convex functions in the sense of symmetric conjugate points is indicated by C_{sc}.

Motivated by the earlier works in Wanas and Majeed in [16], by using quasi-subordinations, we introduce new subfamilies of bi-univalent functions family δ namely, subfamilies of bi-univalent functions in the sense of symmetric conjugate points.

We now define the following:

Definition 1.1. For $0 < \alpha \leq 1$, a function $s \in \delta$ given by (1.1) is said to be in the family $\mathcal{F}_{\delta,q}^{sc}(\alpha, \Xi)$, if it provides the following quasi-subordinations:

$$\frac{2zs'(z)}{s(z) - \overline{s(-\overline{z})}} + \frac{2(zs'(z))'}{(s(z) - \overline{s(-\overline{z})})'} - \frac{2\alpha z^2 s''(z) + 2zs'(z)}{\alpha z(s(z) - \overline{s(-\overline{z})})' + (1-\alpha)(s(z) - \overline{s(-\overline{z})})} - 1 \prec_q (\Xi(z) - 1),$$

and

$$\frac{2wr'(w)}{g(w) - \overline{r(-w)}} + \frac{2(wr'(w))'}{(r(w) - \overline{r(-\overline{w})})'} - \frac{2\alpha w^2 r''(w) + 2wr'(w)}{\alpha w(r(w) - \overline{r(-\overline{w})})' + (1-\alpha)(r(w) - \overline{r(-\overline{w})})} - 1 \prec_q (\Xi(w) - 1),$$

where the function g is the extension of s^{-1} to Ω.

We consider that that for $t(z) = 1$, we have the family $\mathcal{F}_{\delta,q}^{sc}(\alpha, \Xi) = \mathcal{F}_{\delta}^{sc}(\alpha, \Xi)$ as defined as follows:

Definition 1.2. For $0 < \alpha \leq 1$, a function $s \in \delta$ given by (1.1) is said to be in the family $\mathcal{F}_{\delta}^{sc}(\alpha, \Xi)$, if the following subordinations hold:

$$\frac{2zs'(z)}{s(z) - \overline{s(-\overline{z})}} + \frac{2(zs'(z))'}{(s(z) - \overline{s(-\overline{z})})'} - \frac{2\alpha z^2 s''(z) + 2zs'(z)}{\alpha z(s(z) - \overline{s(-\overline{z})})' + (1-\alpha)(s(z) - \overline{s(-\overline{z})})} \prec \Xi(z),$$

and

$$\frac{2wr'(w)}{g(w) - \overline{r(-w)}} + \frac{2(wr'(w))'}{(r(w) - \overline{r(-\overline{w})})'} - \frac{2\alpha w^2 r''(w) + 2wr'(w)}{\alpha w(r(w) - \overline{r(-\overline{w})})' + (1-\alpha)(r(w) - \overline{r(-\overline{w})})} \prec \Xi(w),$$

where the function r is the extension of s^{-1} to Ω.

Upon taking$\alpha = 0$, we obtain the family $\mathcal{F}_{\delta,q}^{sc}(\alpha,\Xi) = \mathcal{F}_{\delta,q}^{sc}(\Xi)$ as defined follow:

Definition 1.3. A functions$\in \delta$ given by (1.1) is said to be in the family $\mathcal{F}_{\delta,q}^{sc}(\Xi)$ if the following subordinations hold:

$$\frac{2(zs'(z))'}{(s(z)-\overline{s(-\overline{z})})'} \prec_q (\Xi(z)-1),$$

and

$$\frac{2(wr'(w))'}{(r(w)-\overline{r(-\overline{w})})'} \prec_q (\Xi(w)-1).$$

Motivated by the earlier works in Wanas and Majeed in [16], we introduce and investigate a new subfamily of functions by the technique of quasi-subordination. The a_2 and a_3 coefficient estimates and also $a_3 - {a_2}^2$ for functions in the subfamily $\mathcal{F}_{\delta,q}^{sc}(\alpha,\Xi)$ are derived. Some important results are also presented in this study.

We should remember here the following lemma so as to derive our basic results:

Lemma 1.4. [5]. If $p \in \mathcal{P}$ then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of functions p analytic in Ω for which $\Re\{p(z)\} > 0$, $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ for $z \in \Omega$.

COEFFICIENT BOUNDS FOR $\mathcal{F}_{\delta,q}^{sc}(\alpha,\Xi)$

In this part, we give resulting estimates for $|a_2|$ and $|a_3|$. Furthermore, $|a_3 - {a_2}^2|$ of the functions $s \in \mathcal{F}_{\delta,q}^{sc}(\alpha,\Xi)$ given by (1) is derived.

We firstly, give main theorem below:

Theorem 2.1. Let $0 < \alpha \leq 1$. If $s \in \mathcal{A}$ of the type (1.1) belongs to the family $\mathcal{F}_{\delta,q}^{sc}(\alpha,\Xi)$, then

$$|a_2| \leq \frac{|t_0|A_1\sqrt{A_1}}{\sqrt{2\left|t_0{A_1}^2(3-2\alpha)-(A_2-A_1)(2-\alpha)^2\right|}}$$

$$|a_3| \leq \frac{|t_1A_1+t_0A_2|}{2(3-2\alpha)},$$

and

$$|a_3-{a_2}^2| \leq \frac{A_1|t_0+t_1|}{2(3-2\alpha)}.$$

Proof. Let $s \in \mathcal{F}_{\delta,q}^{sc}(\alpha,\Xi)$. In the light of Definition 1.1, there are analytic functions $\xi, \phi\colon \Omega \to \Omega$ with $\xi(0) = \phi(0) = 0$, holding

$$\frac{2zs'(z)}{s(z)-\overline{s(-\overline{z})}} + \frac{2(zs'(z))'}{(s(z)-\overline{s(-\overline{z})})'} - \frac{2\alpha z^2 s''(z)+2zs'(z)}{\alpha z(s(z)-\overline{s(-\overline{z})})'+(1-\alpha)(s(z)-\overline{s(-\overline{z})})} - 1 \prec t(z)\left[\Xi(\xi(z))-1\right], \tag{5}$$

and

$$\frac{2wr'(w)}{g(w)-\overline{r(-w)}} + \frac{2(wr'(w))'}{(r(w)-\overline{r(-\overline{w})})'} - \frac{2\alpha w^2 r''(w)+2wr'(w)}{\alpha w(r(w)-\overline{r(-\overline{w})})'+(1-\alpha)(r(w)-\overline{r(-\overline{w})})} - 1 \prec t(w)[\Xi(\phi(w))-1]. \tag{6}$$

Define the functions

$$\eta(z) = \frac{1+\xi(z)}{1-\xi(z)} = 1 + \xi_1 z + \xi_2 z^2 + \xi_3 z^3 + \cdots,$$

and

$$\mu(z) = \frac{1+\phi(z)}{1-\phi(z)} = 1 + \phi_1 z + \phi_2 z^2 + \phi_3 z^3 + \cdots$$

or equivalently,

$$\xi(z) = \frac{\eta(z)-1}{\eta(z)+1} = \frac{1}{2}\left[\xi_1 z + \left(\xi_2 - \frac{{\xi_1}^2}{2}\right)z^2 + \cdots\right],$$

and

$$\phi(w) = \frac{\mu(w)-1}{\mu(w)+1} = \frac{1}{2}\left[\phi_1 w + \left(\phi_2 - \frac{{\phi_1}^2}{2}\right)w^2 + \cdots\right].$$

Then, $\eta(z)$ and $\mu(w)$ analytic in Ω with $\eta(0) = \mu(0) = 1$. Since the functions $\eta(z)$ and $\mu(w)$ have a positive real part in Ω, $|\xi_i| \leq 2$ and $|\phi_i| \leq 2$. Now,

$$t(z)[\Xi(\xi(z))-1] = \frac{1}{2}t_0A_1\xi_1 z + \left[\frac{1}{2}t_1A_1\xi_1 + \frac{1}{2}t_0A_1\left(\xi_2 - \frac{1}{2}{\xi_1}^2\right) + \frac{1}{4}t_0A_2{\xi_1}^2\right]z^2 + \cdots, \tag{7}$$

and

$$t(w)[\Xi(\phi(w))-1] = \frac{1}{2}t_0A_1\phi_1 w + \left[\frac{1}{2}t_1A_1\phi_1 + \frac{1}{2}t_0A_1\left(\phi_2 - \frac{1}{2}{\phi_1}^2\right) + \frac{1}{4}t_0A_2{\phi_1}^2\right]w^2 + \cdots. \tag{8}$$

In the view of (5), (6) and (7), (8) we obtain

$$2(2-\alpha)a_2 = \frac{1}{2}t_0A_1\xi_1\,, \tag{9}$$

$$2(3-2\alpha)a_3 = \frac{1}{2}t_1A_1\xi_1 + \frac{1}{2}t_0A_1\left(\xi_2 - \frac{1}{2}{\xi_1}^2\right) + \frac{1}{4}t_0A_2{\xi_1}^2, \quad (10)$$

$$-2(2-\alpha)a_2 = \frac{1}{2}t_0A_1\phi_1\,, \quad (11)$$

$$2(3-2\alpha)(2{a_2}^2 - a_3) = \frac{1}{2}t_1A_1\phi_1 + \frac{1}{2}t_0A_1\left(\phi_2 - \frac{1}{2}{\phi_1}^2\right) + \frac{1}{4}t_0A_2{\phi_1}^2. \quad (12)$$

It follows from (9) and (11) that

$$\xi_1 = -\phi_1, \quad (13)$$

and

$$16(2-\alpha)^2{a_2}^2 = {t_0}^2{A_1}^2({\xi_1}^2 + {\phi_1}^2). \quad (14)$$

Now, by adding (10) and (12), also using (13), we get,

$$4(3-2\alpha){a_2}^2 = \frac{1}{2}t_0A_1(\xi_2+\phi_2) + \frac{1}{4}t_0(A_2-A_1)({\xi_1}^2 + {\phi_1}^2). \quad (15)$$

Using (14) in (15), we obtain

$${a_2}^2 = \frac{{t_0}^2{A_1}^3(\xi_2+\phi_2)}{8\left[t_0{A_1}^2(3-2\alpha) - (A_2-A_1)(2-\alpha)^2\right]}.$$

Applying $|\xi_i| \le 2$ and $|\phi_i| \le 2$ for the coefficients ξ_2 and ϕ_2, we immediately get

$$|{a_2}^2| \le \frac{{t_0}^2{A_1}^3}{2\left|t_0{A_1}^2(3-2\alpha) - (A_2-A_1)(2-\alpha)^2\right|}.$$

This gives the desired bound on $|a_2|$ as given in Theorem 2.1.

In addition to calculate the third bound, by subtracting (12) from (10), we immediately have the equality given below:

$$4(3-2\alpha)(a_3 - {a_2}^2) = \frac{1}{2}t_1A_1(\xi_1-\phi_1) + \frac{1}{2}t_0A_1(\xi_2-\phi_2) + \frac{1}{4}t_0(A_2-A_1)({\xi_1}^2 + {\phi_1}^2). \quad (16)$$

Using (15) in (16), we get

$$a_3 = \frac{t_1A_1(\xi_1-\phi_1)}{8(3-2\alpha)} + \frac{t_0A_1\xi_2}{4(3-2\alpha)} + \frac{t_0(A_2-A_1){\xi_1}^2}{8(3-2\alpha)}.$$

Applying $|\xi_i| \le 2$ and $|\phi_i| \le 2$ for the coefficients ξ_1, ξ_2 and ϕ_1, ϕ_2, we obtain the desired bound on $|a_3|$ given in Theorem 2.1.

Lastly, from (16) and (13)

$$a_3 - {a_2}^2 = \frac{t_1A_1(\xi_1-\phi_1)}{8(3-2\alpha)} + \frac{t_0A_1(\xi_2-\phi_2)}{8(3-2\alpha)}.$$

Applying again $|\xi_i| \le 2$ and $|\phi_i| \le 2$ over again for the coefficients ξ_1, ξ_2 and ϕ_1, ϕ_2, we can easily acquire the last desired result in Theorem 2.1.

CONCLUDING COROLLARIES

For the function Ξ is given by

$$\Xi(z) = \left(\frac{1+z}{1-z}\right)^n = 1 + 2nz + 2n^2z^2 + \cdots \qquad (0 < n \le 1),$$

which gives

$$A_1 = 2n \quad \text{and} \quad A_2 = 2n^2,$$

Theorem 2.1 reduces to the following corollary.

Corollary 3.1. Let $s \in \mathcal{F}^{sc}_{\delta,q}\left[\alpha, \left(\frac{1+z}{1-z}\right)^n\right]$. Then

$$|a_2| \le \frac{2|t_0|n}{\sqrt{2|n[4(1-t_0)\alpha + 2(3t_0-2) - \alpha^2] + (\alpha-2)^2|}},$$

$$|a_3| \le \frac{n|t_1 + nt_0|}{(3-2\alpha)},$$

and

$$a_3 - {a_2}^2| \le \frac{n|t_0 + t_1|}{(3-2\alpha)}.$$

Remark 3.2. For $\alpha = 0$ and $s \in \mathcal{F}^{sc}_{\delta,q}\left[\left(\frac{1+z}{1-z}\right)^n\right]$, the results in Corollary 3.1 reduced to results given below.

Upon considering $-1 \le A \le B < 1$ and the function $\Xi(z)$ given by

$$\Xi(z) = \frac{1+Bz}{1+Az} = 1 + (B-A)z - A(B-A)z^2 + \cdots.$$

Then we get

$$A_1 = B - A \ \text{ and } \ A_2 = -A(B-A).$$

So, the following corollary can be obtain readily,

Corollary 3.3. Let $s \in \mathcal{F}^{sc}_{\delta,q}\left(\alpha, \frac{1+Bz}{1+Az}\right)$. Then

$$|a_2| \le \frac{|t_0|(B-A)}{\sqrt{2|t_0(B-A)(3-2\alpha) - (A+1)(2-\alpha)^2|}},$$

$$|a_3| \le \frac{(B-A)|(t_1 - t_0A)|}{2(3-2\alpha)},$$

and

$$|a_3 - a_2{}^2| \le \frac{(B-A)|t_0 + t_1|}{2(3-2\alpha)}.$$

Remark 3.4. Upon taking $\alpha = 0$ in Corollary 3.3, we immediately acquire the following results:

$$|a_2| \le \frac{|t_0|(B-A)}{\sqrt{2|3t_0(B-A) - 4(A+1)|}},$$

$$|a_3| \le \frac{(B-A)|(t_1 - t_0 A)|}{6},$$

and

$$|a_3 - a_2{}^2| \le \frac{(B-A)|t_0 + t_1|}{6}.$$

Furthermore when the function $\Xi(\mathrm{z})$ is given by

$$\Xi(\mathrm{z}) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \le \beta < 1),$$

we have

$$A_1 = A_2 = 2(1-\beta).$$

Hence, we reach the corollary given below.

Corollary 3.5. Let $s \in \mathcal{F}_{\delta,q}^{sc}\left(\alpha, \frac{1+(1-2\beta)z}{1-z}\right)$, Then

$$|a_2| \le \frac{|t_0|\sqrt{1-\beta}}{\sqrt{|t_0(3-2\alpha)|}},$$

$$a_3| = |a_3 - a_2{}^2| \le \frac{(1-\beta)|t_0 + t_1|}{(3-2\alpha)}.$$

Remark 3.6. Upon taking $\alpha = 0$ in Corollary 3.5, we can immediately acquire the following results

$$|a_2| \le \frac{|t_0|\sqrt{1-\beta}}{\sqrt{3|t_0|}},$$

$$a_3| = |a_3 - a_2{}^2| \le \frac{(1-\beta)|t_0 + t_1|}{3}.$$

Finally, considering the function $\Xi(\mathrm{z})$ as follows:

$$\Xi(\mathrm{z}) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots,$$

we obtain

$$A_1 = A_2 = 2.$$

So, we acquire the corollary given below.

Corollary 3.7. Let $s \in \mathcal{F}_{\delta,q}^{sc}\left(\alpha, \frac{1+z}{1-z}\right)$. Then

$$|a_2| \le \frac{|t_0|}{\sqrt{|t_0(3-2\alpha)|}},$$

$$|a_3| = |a_3 - a_2{}^2| \le \frac{|t_0 + t_1|}{(3-2\alpha)}.$$

Remark 3.8. Upon taking $\alpha = 0$ in Corollary 3.7, we readily reach that

$$|a_2| \le \frac{|t_0|}{\sqrt{3|t_0|}},$$

and

$$|a_3| = |a_3 - a_2{}^2| \le \frac{|t_0 + t_1|}{3}.$$

Remark 3.9. For the case of $t(z) \equiv 1$ and different versions of $\Xi(\mathrm{z})$, other different results can be obtained. The details of these results may be left as an exercise for the interested readers.

CONCLUSION

Motivated by the earlier works in Wanas and Majeed in [16], we have introduced and investigated a new subfamily of functions by the technique of quasi-subordination. Then, our main conclusion is expressed as Theorem 2.1 and proved. The second and third Maclaurin-coefficients $|a_2|$, $|a_3|$ and also related with these coefficients $|a_3 - a_2{}^2|$ for functions belonging to this subfamily have derived. In addition, new corollaries and remarks have been obtained for different selections of some parameters.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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