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On the Hankel Determinant of *m*-fold Symmetric Bi-Univalent Functions Using a New Operator

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Highlights

- This paper focuses on definition of a new subclass of bi-univalent functions.
- The convolution is proposed for construction of the operator $\mathcal{J}_{s,a,\mu}^{\delta,\lambda} f(z)$.
- The second Hankel determinant were obtained.

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Abstract

In this article, we aim to describe a new operator $\mathcal{J}_{s,a,\mu}^{\delta,\lambda}$ via convolution. Moreover, we aim to present a new subclass $\boldsymbol{\mathcal{C}}_{\Sigma m}(\tau;\beta)$ related to *m*-fold symmetric bi-univalent functions in the open unit disk $\Theta = \{z \in \mathbb{C} : |z| < 1\}$. Finally, an estimate related to the Hankel determinant for functions in $\boldsymbol{\mathcal{C}}_{\Sigma m}(\tau;\beta)$ are given.

1. INTRODUCTION

Assume $\mathcal{A}$ is the class of normalized analytic functions in Θ with Taylor series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Assume $\mathcal{S}$ is the class of univalent functions from $\mathcal{A}$ in Θ. Further, every function $f \in \mathcal{S}$ has an inverse f^{-1} fulfilling $f^{-1}\big(f(z)\big) = z \ (z \in \Theta)$ and $f(f^{-1}(w)) = w \ \big(w \in \Theta_p\big)$, where $p \geq \frac{1}{4}$ denotes the radius of the image $f(\Theta)$ and $\Theta_p = \{z \in \mathbb{C} : |z| < p\}$ [1]. It is recalled that

$$g(w) = f^{-1}(w) = w + C_2 w^2 + C_3 w^3 + C_4 w^4 + \cdots, \tag{2}$$

where

$$C_2 = -a_2$$
$$C_3 = (2a_2^2 - a_3)$$
$$C_4 = -(5a_2^3 - 5a_2 a_3 + a_4).$$

A function $f \in \mathcal{A}$ is named bi-univalent in Θ if both f and f^{-1} are univalent in Θ. Next, assume $\sum$ is the class of bi-univalent functions $f \in \mathcal{A}$ in Θ. For a detailed literature and fundamental examples of $\sum$, see the leading paper by Srivastava *et al.* [2] (see also [3-10]).

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Assume $f \in \mathcal{A}$ and g is given by $g(z) = z + \sum_{k=2}^{\infty} b_k\ z^k$. The convolution (Hadamard product) of f and g is represented by $(f * g)(z)$ and expressed by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k\, z^k.$$

Aldweby and Darus [11] established the Ruscheweyh type q-analogue operator $\mathcal{R}_q^{\delta}$ by

$$\mathcal{R}_q^{\delta} f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\delta-1]_q!}{[\delta]_q!\ [k-1]_q!}\, a_k\, z^k\,,$$

where $\delta \geq 0$. Also, as $q \to 1^-$, we have

$$\begin{aligned}\lim_{q \to 1^-} \mathcal{R}_q^{\delta} f(z) &= z + \lim_{q \to 1^-} \left[\sum_{k=2}^{\infty} \frac{[k+\delta-1]_q!}{[\delta]_q!\ [k-1]_q!}\, a_k\, z^k\right] \\ &= z + \sum_{k=2}^{\infty} \frac{(k+\delta-1)!}{(\delta)!(k-1)!}\, a_k\, z^k \\ &= \mathcal{R}_q^{\delta} f(z)\,.\end{aligned}$$

Komatu [12] introduced a family of integral operator $J_{\mu}^{\lambda} : \Sigma \to \Sigma$ by

$$J_{\mu}^{\lambda} f(z) = z + \sum_{k=1}^{\infty} \left(\frac{\mu}{\mu+k-1}\right)^{\lambda} a_k\, z^k, \qquad (z \in U^* = \Theta \backslash \{0\}, k > 1, \lambda \geq 0).$$

By using the Hurwitz - Lerch Zeta function

$$\emptyset(z,s,a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \ \ (a \in \mathbb{R}/z_0^-, s \in \mathbb{R}\ \ when\ \ 0 < |z| < 1)$$

and $G_{s,a(z)}$ is given by

$$G_{s,a(z)} = (1+a)^s\, [\,\emptyset(z,s,a) - a^{-s}\,],$$

the linear operator $I_{s,a,\mu}^{\lambda}\ f(z) : \Sigma \to \Sigma$ is expressed by [13]

$$I_{s,a,\mu}^{\lambda} f(z) = G_{s,a(z)} * J_{\mu}^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+a}{t+a}\right)^s \left(\frac{\mu}{\mu+k-1}\right)^{\lambda} a_k\, z^k\,.$$

The convolution of the operators $\mathcal{R}_q^{\delta} f(z)$ and $I_{s,a,\mu}^{\lambda} f(z)$ defined as

$$\mathcal{J}_{s,a,\mu}^{\delta,\lambda} f(z) = \mathcal{R}_q^{\delta} f(z) * I_{s,a,\mu}^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{(k+\delta-1)!}{(\delta)!\,(k-1)!}\right)\left(\frac{1+a}{t+a}\right)^s \left(\frac{\mu}{\mu+k-1}\right)^{\lambda} a_k z^k$$

$$\mathcal{J}_{s,a,\mu}^{\delta,\lambda} f(z) = z + \sum_{k=2}^{\infty} \theta_{k,\delta}\ a_k\, z^k,$$

where $\theta_{k,\delta} = \left\{\left(\frac{(k+\delta-1)!}{(\delta)!(k-1)!}\right)\left(\frac{1+a}{t+a}\right)^s \left(\frac{\mu}{\mu+k-1}\right)^{\lambda}\right\}$.

Now, for a function $f \in \mathcal{S}$, the function $h(z) = \left(f(z^m)\right)^{\frac{1}{m}}$ $(z \in \Theta,\ m \in \mathbb{N})$ is univalent. Further, that function maps Θ into an m-fold symmetry region. Thus, a function is named m-fold symmetric ([14,15]) if it has the normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in \Theta,\ m \in \mathbb{N}). \tag{3}$$

Assume $\mathcal{S}_m$ is the class of m-fold symmetric univalent functions given by (3). We inform that the functions in $\mathcal{S}$ are one-fold symmetric. Similar to the definition of m-fold symmetric univalent functions, m-fold symmetric bi-univalent functions are presented. The series expansion of f^{-1} is expressed by [16]

$$g(w) = w - a_{m+1} w^{m+1} + [\,(m+1)\, a_{m+1}^2 - a_{2m+1}\,)]\, w^{2m+1} - [\,\frac{1}{2}(m+1)\,(3m+2)a_{m+1}^3$$
$$-(3m+2)\, a_{m+1}\, a_{2m+1} + a_{3m+1}\,]\, w^{3m+1} + \cdots, \tag{4}$$

where $f^{-1} = g$. Assume $\sum_m$ is the class of m-fold symmetric bi-univalent functions in Θ. If we set $m = 1$, the formula (4) reduces to the formula (2) of the class $\sum$.

Next, Noonan and Thomas [17] introduced the q^{th} Hankel determinant of f by

$$H_q\,(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \qquad (a_1 = 1,\ n \geq 0,\ q \geq 1).$$

Note that

$$H_2\,(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \quad H_2\,(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$$

and $H_2(1)$ is known as Fekete-Szegö functional (see [18]). After that, this approach has been studied by several researchers ([19-23]).

2. MATERIAL METHOD

To present our outcomes, we must recall some lemmas.

Lemma 2.1. [15] Assume $\mathcal{P}$ is class of functions p analytic in Θ for which $Re(p(z)) > 0$. If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, then $|\,p_i| \leq 2$ for each $i \in \mathbb{N}$.

Lemma 2.2. [24] If $p \in \mathcal{P}$, then

$$2p_2 = p_1^2 + (\,4 - p_1^2\,)x$$
$$4p_3 = p_1^3 + 2(\,4 - p_1^2\,)p_1 x - (4 - p_1^2)p_1 x^2 + 2(\,4 - p_1^2\,)(1 - |x|^2)z$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Now, we shall introduce the class $\boldsymbol{C}_{\Sigma\mathrm{m}}\,(\tau; \beta)$ as follows.

Definition 3.1. A function $f \in \sum$ is in $\boldsymbol{C}_{\Sigma\mathrm{m}}\,(\tau; \beta)$ $(z, w \in \Theta,\ 0 \leq \tau \leq 1,\ 0 \leq \beta < 1,\ m \in \mathbb{N})$ if it fulfills

$$Re\left[(2\tau+1)\,\frac{z\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda}\,f(z)\right)'}{z}-\tau z\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda}\,f(z)\right)''-2\tau\right]>\beta$$

and

$$Re\left[(2\tau+1)\,\frac{w\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda}\,g(w)\right)'}{w}-\tau w\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda}\,g(w)\right)''-2\tau\right]>\beta,$$

where the function g is the extension of f^{-1} to Θ.

3. THE RESEARCH FINDINGS AND DISCUSSION

Theorem 3.1. Assume f is in the class $\boldsymbol{C}_{\Sigma\mathrm{m}}\,(\tau\,;\beta)$. For $0\leq\tau\leq1$, $0\leq\beta<1$, we find

$$\left|\,a_{m+1}a_{3m+1}-a_{2m+1}^2\right|$$

$$\leq\begin{cases}E(\beta,2-), & if\ \Phi(\beta,p)\geq0 \text{ and } \vartheta(\beta,p)\geq0\\ \dfrac{4(1-\beta)^2}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}, & if\ \Phi(\beta,p)\leq0 \text{ and } \vartheta(\beta,p)\leq0\\ \max\left\{\dfrac{4(1-\beta)^2}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2},E(\beta,2-)\right\}, & if\ \Phi(\beta,p)>0 \text{ and } \vartheta(\beta,p)<0\\ \max\{E(\beta,p_0),E(\beta,2-)\}, & if\ \Phi(\beta,p)<0 \text{ and } \vartheta(\beta,p)>0,\end{cases}$$

where

$$E(\beta,2-)=\frac{4(1-\beta)^2}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}+(1-\beta)^2\,[8\Phi(\beta,p)+2\vartheta(\beta,p)],$$

$$E(\beta,p_0)=\frac{4\,(1-\beta)^2}{(2m+1)^2\,[2\tau(1-m)+1]^2\,\theta_{2m+1,\delta}^2}-\frac{(1-\beta)^2\,\vartheta^2(\beta,p)}{8\,\Phi(\beta,p)},$$

$$p_0=\sqrt{\frac{-\vartheta(\beta,p)}{2\,\Phi(\beta,p)}},$$

$$\Phi(\beta,p)=\frac{(1-\beta)^2}{2(m+1)^2[\tau(2-m)+1]^4\theta_{m+1,\delta}^4}-\frac{(3m+2)(1-\beta)}{2(m+1)^2(m+1)[\tau(2-m)+1]^2[2\tau(1-m)+1]}$$
$$-\frac{1}{(m+1)(3m+1)[\tau(2-3m)+1][\tau(2-m)+1]\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}$$
$$+\frac{1}{2(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}$$

and

$$\vartheta(\beta,p)=\frac{2(3m+2)(1-\beta)}{(m+1)^2(2m+1)[\tau(2-m)+1]^2[2\tau(1-m)+1]}$$
$$+\frac{6}{(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\theta_{3m+1,\delta}}$$
$$-\frac{4}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}.$$

Proof. Let $f \in \mathcal{C}_{\Sigma\mathrm{m}}(\tau;\beta)$. Then

$$(2\tau+1)\,\frac{z\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda} f(z)\right)'}{z} - \tau\, z\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda} f(z)\right)'' - 2\tau = \beta + (1-\beta)\,p(z), \tag{5}$$

$$(2\tau+1)\,\frac{w\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda} g(w)\right)'}{w} - \tau\, w\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda} g(w)\right)'' - 2\tau = \beta + (1-\beta)\,q(w), \tag{6}$$

where $g = f^{-1}$ and

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots,$$
$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots.$$

If we equate the coefficients in (5) and (6), we find

$$[\tau(2-m)+1](m+1)\,\theta_{m+1,\delta}\,a_{m+1} = (1-\beta)\,p_m, \tag{7}$$

$$[2\tau(1-m)+1](2m+1)\,\theta_{2m+1,\delta}\,a_{2m+1} = (1-\beta)\,p_{2m}, \tag{8}$$

$$[\tau(2-3m)+1](3m+1)\,\theta_{3m+1,\delta}\,a_{3m+1} = (1-\beta)\,p_{3m}, \tag{9}$$

$$-[\tau(2-m)+1](m+1)\,\theta_{m+1,\delta}\,a_{m+1} = (1-\beta)\,q_m, \tag{10}$$

$$[2\tau(1-m)+1]\,[(m+1)\,a_{m+1}^2 - a_{2m+1}](2m+1)\,\theta_{2m+1,\delta} = (1-\beta)\,q_{2m}, \tag{11}$$

$$-[\,\tau\,(2-3m)+1]\left[\frac{1}{2}\,(m+1)\,(3m+2)\,a_{m+1}^3 - (3m+2)\,a_{m+1}\,a_{2m+1} + a_{3m+1}\right]$$
$$(3m+1)\theta_{3m+1,\delta} = (1-\beta)\,q_{3m}\,. \tag{12}$$

From (7) and (10), we get

$$p_m = -q_m, \tag{13}$$
$$a_{m+1} = \frac{(1-\beta)}{(m+1)[\tau(2-m)+1]\,\theta_{m+1,\delta}}\,p_m.$$

Subtracting (8) from (11), we have

$$a_{2m+1} = \frac{(1-\beta)^2}{2(m+1)[\tau(2-m)+1]^2\,\theta_{m+1,\delta}^2}\,p_m^2 + \frac{(1-\beta)(\,p_{2m}-q_{2m}\,)}{2(2m+1)\,[2\tau(1-m)+1]\,\theta_{2m+1,\delta}}\,.$$

Also, subtracting (9) from (12), we obtain

$$a_{3m+1} = \frac{(3m+2)\,(1-\beta)^2 p_m(p_{2m}-q_{2m})}{4\,(m+1)\,(2m+1)\,[\tau(2-m)+1]\,[2\tau(1-m)+1]\,\theta_{m+1,\delta}\,\theta_{2m+1,\delta}}$$
$$+\frac{(1-\beta)(\,p_{3m}-q_{3m}\,)}{2(3m+1)[\tau(2-3m)+1]\theta_{3m+1,\delta}}\,.$$

Then, we can establish that

$$|\,a_{m+1}a_{3m+1} - a_{2m+1}^2| = \left|-\frac{(1-\beta)^4}{4\,(m+1)^2\,[\tau(2-m)+1]^4\theta_{m+1,\delta}^4}p_m^4\right.$$

$$+\frac{(3m+2)\,(1-\beta)^3}{4\,(m+1)^2\,(2m+1)[\tau(2-m)+1]^2\,[2\tau(1-m)+1]\,\theta_{m+1,\delta}}\,p_m^2(\,p_{2m}-q_{2m}\,)$$
$$+\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}\,p_m(\,p_{3m}-q_{3m}\,)$$
$$-\frac{(1-\beta)^2}{4(2m+1)^2\,[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}\,\left(\,p_{2m}-q_{2m}\,\right)^2\Big|. \tag{14}$$

According to Lemma (2.2) and in light of (13), we write

$$\left.\begin{aligned}2p_{2m}&=p_m^2+(\,4-p_m^2\,)x\\ 2q_{2m}&=q_m^2+(\,4-q_m^2\,)y\end{aligned}\right\}\implies p_{2m}-q_{2m}=\frac{4-p_m^2}{2}\,(x-y), \tag{15}$$

$$4p_{3m}=p_m^3+2(\,4-p_m^2\,)p_m x-p_m(4-p_m^2)x^2+2(\,4-p_m^2\,)(1-|x|^2)z,$$
$$4q_{3m}=q_m^3+2(\,4-q_m^2\,)q_m y-q_m(4-q_m^2)y^2+2(\,4-q_m^2\,)(1-|y|^2)w,$$

$$p_{3m}-q_{3m}=\frac{p_m^3}{2}+\frac{p_m\,(\,4-p_m^2\,)}{2}\,(x+y)-\frac{p_m\,(\,4-p_m^2\,)}{4}\,(x^2+y^2)$$
$$+\frac{4-p_m^2}{2}\,[\,(1-|x|^2)\,z\,-\,(1-|y|^2)\,w\,]\,. \tag{16}$$

Then, using (15) and (16) in (14), we get

$$|\,a_{m+1}a_{3m+1}-a_{2m+1}^2| = \Bigg|-\frac{(1-\beta)^4}{4\,(m+1)^2\,[\tau(2-m)+1]^4\theta_{m+1,\delta}^4}p_m^4$$
$$+\frac{(3m+2)\,(1-\beta)^3}{4\,(m+1)^2\,(2m+1)[\tau(2-m)+1]^2\,[2\tau(1-m)+1]}\;p_m^2\,\frac{(4-p_m^2)}{2}\,(x-y)$$
$$+\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}\,\frac{p_m^4}{2}$$
$$+\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}p_m^2\,\frac{4-p_m^2}{2}(x+y)$$
$$-\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}\,p_m^2\,\frac{(4-p_m^2)}{4}(x^2+y^2)$$
$$+\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}\,p_m$$
$$\times\frac{(4-p_m^2)}{2}\,[(1-|x|^2)z-(1-|y|^2)w]-\frac{(1-\beta)^2}{4(2m+1)^2\,[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}$$
$$\times\frac{\left(4-p_m^2\right)^2}{4}\,(x-y)^2\,|. \tag{17}$$

$$\left|a_{m+1}a_{3m+1}-a_{2m+1}^2\right|\le\frac{(1-\beta)^4}{4\,(m+1)^2\,[\tau(2-m)+1]^4\,\theta_{m+1,\delta}^4}\,p_m^4$$
$$+\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1]\,[\tau(2-3m)+1]\,\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}\,\frac{p_m^4}{2}$$
$$+\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1]\,[\tau(2-3m)+1]\,\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}\,p_m\,(4-p_m^2)$$
$$+(|x|+|y|)\left[\frac{(3m+2)(1-\beta)^3}{4(m+1)^2(2m+1)[\tau(2-m)+1]^2[2\tau(1-m)+1]}p_m^2\,\frac{(4-p_m^2)}{2}\right.$$
$$\left.+\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\theta_{3m+1,\delta}}p_m^2\,\frac{(4-p_m^2)}{2}\right]$$

$$+(|x|^2+|y|^2)\left[\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\theta_{3m+1,\delta}}p_m^2\frac{(4-p_m^2)}{4}\right.$$
$$\left.-\frac{(1-\beta)^2}{2(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\theta_{3m+1,\delta}}p_m\frac{(4-p_m^2)}{4}\right]$$

$$+\frac{(1-\beta)^2}{4(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}\frac{(4-p_m^2)^2}{4}(|x|+|y|)^2\,.$$

Since $p\in P,\ |p_m|\le 2$. Letting $|p_m|=p$, we may assume without restriction that $p\in[0,2]$. For $\gamma=|x|\le 1\ \ and\ \alpha=|y|\le 1$, we get

$$\left|a_{m+1}a_{3m+1}-a_{2m+1}^2\right|\le A_1+(\gamma+\alpha)A_2+(\gamma^2+\alpha^2)A_3+(\gamma+\alpha)^2A_4=H(\gamma,\alpha),$$

where

$$A_1=A_1(\beta,p)=\frac{(1-\beta)^2}{2(m+1)[\tau(2-m)+1]\theta_{m+1,\delta}}\left[\left(\frac{(1-\beta)^2}{2(m+1)[\tau(2-m)+1]^3\theta_{m+1,\delta}^3}\right.\right.$$
$$\left.+\frac{1}{2(3m+1)[\tau(2-3m)+1]\theta_{3m+1,\delta}}\right)p^4-\frac{1}{(3m+1)[\tau(2-3m)+1]\theta_{3m+1,\delta}}p^3$$
$$\left.+\frac{4}{(3m+1)[\tau(2-3m)+1]\theta_{3m+1,\delta}}p\right]\ge 0,$$

$$A_2=A_2(\beta,p)=\frac{(1-\beta)^2}{4(m+1)[\tau(2-m)+1]}\ p^2(4-\ p^2)$$
$$\left[\frac{(3m+2)(1-\beta)}{2(m+1)(2m+1)[\tau(2-m)+1][2\tau(1-m)+1]}\right.$$
$$\left.+\ \frac{1}{(3m+1)[\tau(2-3m)+1]\theta_{m+1,\delta}\theta_{3m+1,\delta}}\right]\ge 0,$$

$$A_3=A_3(\beta,p)=\frac{(1-\beta)^2}{8(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\theta_{3m+1,\delta}},$$

$$p(4-p^2)(p-2)\le 0$$

$$A_4=A_4(\beta,p)=\frac{(1-\beta)^2}{4(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}\ \frac{(4-p^2)^2}{4}\ge 0.$$

Now, we aim to maximize $H(\gamma,\alpha)$ on the square $[0,1]\times[0,1]$. Hence, we need the maximum of $H(\gamma,\alpha)$ for the cases $p\in(0,2), p=0\ and\ p=2$. If $p\in(0,2)$, we find $A_3<0$ and $A_3\ +2A_4>0$. Thus, we conclude that

$$H_{\gamma\gamma}\,.\,H_{\alpha\alpha}-\left(H_{\gamma\alpha}\right)^2<0$$

which means that H cannot have a local maximum in the interior of the square. Hence, we examine the boundary of the square.

For $\gamma=0\ and\ 0\le\alpha\le 1$ (similarly $\alpha=0\ and\ 0\le\gamma\le 1$), we have

$$H(0,\alpha)=N(\alpha)=(A_3+A_4)\alpha^2+A_2\alpha+A_1.$$

1- The case $A_3 + A_4 \geq 0$: For $0 < \alpha < 1$ and any fixed p $(0 < p < 2)$, we find that $N'(\alpha) = 2(A_3 + A_4)\alpha + A_2 > 0$, that is, $N(\alpha)$ is increasing. Thus, the maximum of $N(\alpha)$ occurs at $\alpha = 1$ and

$$max\, N(\alpha) = N(1) = A_1 + A_2 + A_3 + A_4.$$

2- The case $A_3 + A_4 < 0$: Since $A_2 + 2(A_3 + A_4) \geq 0$, we find that $A_2 + 2(A_3 + A_4) < 2(A_3 + A_4)\alpha + A_2 < A_2$ and so $N'(\alpha) > 0$. Thus, the maximum of $N(\alpha)$ occurs at $\alpha = 1$.

If $p = 2$, we get

$$H(\gamma, \alpha) = \frac{(1-\beta)^2}{(m+1)[\tau(2-m)+1]\theta_{m+1,\delta}} \times \left[\frac{4(1-\beta)^2}{(m+1)[\tau(2-m)+1]^3\theta_{m+1,\delta}^3} + \frac{4}{(3m+1)[\tau(2-3m)+1]\theta_{3m+1,\delta}}\right]. \tag{18}$$

From (18) and the above cases, we arrive

$$max\, N(\alpha) = N(1) = A_1 + A_2 + A_3 + A_4 \quad (0 \leq \alpha \leq 1,\ \ 0 \leq p \leq 2).$$

For $\gamma = 1$ and $0 \leq \alpha \leq 1$ (similarly $\alpha = 1$ and $0 \leq \gamma \leq 1$), we obtain

$$H(1, \alpha) = L(\alpha) = (A_3 + A_4)\alpha^2 + (A_2 + 2A_4)\alpha + A_1 + A_2 + A_3 + A_4.$$

Similar to the above cases of $A_3 + A_4$, we have

$$max\, L(\alpha) = L(1) = A_1 + 2A_2 + 2A_3 + 4A_4.$$

Since $N(1) \leq L(1) for\ p \in [0,2]$, $max\, H(\gamma, \alpha) = H(1,1)$. Therefore, the maximum of H occurs at $\gamma = 1$ and $\alpha = 1$ in the closed square.

Let $E: [0,2] \to R$

$$E(\beta, p) = max\, H(\gamma, \alpha) = H(1,1) = A_1 + 2A_2 + 2A_3 + 4A_4. \tag{19}$$

Substituting the values of A_1, A_2, A_3 and A_4 from (19) yields

$$\begin{aligned} E(\beta, p) = \frac{(1-\beta)^2}{2}\Bigg[\Bigg(&\frac{(1-\beta)^2}{2(m+1)^2[\tau(2-m)+1]^4\theta_{m+1,\delta}^4} \\ &- \frac{(3m+2)(1-\beta)}{2(m+1)^2(2m+1)[\tau(2-m)+1]^2[2\tau(1-m)+1]} \\ &- \frac{1}{(m+1)(3m+1)[\tau(2-3m)+1][\tau(2-m)+1]\theta_{m+1,\delta}\,\theta_{3m+1,\delta}} \\ &+ \frac{1}{2(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}\Bigg)p^4 \\ &+ \Bigg(\frac{2(3m+2)(1-\beta)}{(m+1)^2(2m+1)[\tau(2-m)+1]^2[2\tau(1-m)+1]} \\ &+ \frac{6}{(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\theta_{3m+1,\delta}} \\ &- \frac{4}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}\Bigg)p^2 + \frac{8}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}\Bigg]. \end{aligned}$$

$$E(\beta,p)=\frac{(1-\beta)^2}{2}\left[\Phi(\beta,p)p^4+\vartheta(\beta,p)p^2\right]+\frac{4(1-\beta)^2}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2},$$

where

$$\begin{aligned}\Phi(\beta,p)=&\frac{(1-\beta)^2}{2(m+1)^2[\tau(2-m)+1]^4\theta_{m+1,\delta}^4}-\frac{(3m+2)(1-\beta)}{2(m+1)^2(m+1)[\tau(2-m)+1]^2[2\tau(1-m)+1]}\\&-\frac{1}{(m+1)(3m+1)[\tau(2-3m)+1][\tau(2-m)+1]\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}\\&+\frac{1}{2(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2},\end{aligned}$$

$$\begin{aligned}\vartheta(\beta,p)=&\frac{2(3m+2)(1-\beta)}{(m+1)^2(2m+1)[\tau(2-m)+1]^2[2\tau(1-m)+1]}\\&+\frac{6}{(m+1)(3m+1)[\tau(2-m)+1][\tau(2-3m)+1]\theta_{m+1,\delta}\theta_{3m+1,\delta}}\\&-\frac{4}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}.\end{aligned}$$

Assume that $E(\beta,p)$ has a maximum value in an interior of $p\in[0,2]$, then

$$E'(\beta,p)=(1-\beta)^2[2\,\Phi(\beta,p)p^3+\vartheta(\beta,p)p].$$

Now, we msut investigate the function $E'(\beta,p)$ due to the different cases of $\Phi(\beta,p)$ and $\vartheta(\beta,p)$.

(i) Let $\Phi(\beta,p)\ge 0$ and $\vartheta(\beta,p)\ge 0$, then $E'(\beta,p)\ge 0$, so $E(\beta,p)$ is an increasing function. Therefore,

$$\begin{aligned}max\{E(\beta,p):p\in(0,2)\}=E(\beta,2-)=&\frac{4(1-\beta)^2}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}\\&+(1-\beta)^2\,[8\Phi(\beta,p)+2\vartheta(\beta,p)],\end{aligned}$$

that is,

$$max\{max\{H(\gamma,\alpha):\gamma,\alpha\in[0,1]\}:p\in(0,2)\}=E(\beta,2-).$$

(ii) Let $\Phi(\beta,p)\le 0$ and $\vartheta(\beta,p)\le 0$, then $E'(\beta,p)\le 0$. Thus, $E(\beta,p)$ is a decreasing. Therefore,

$$max\{E(\beta,p):p\in(0,2)\}=E(\beta,0+)=\frac{4(1-\beta)^2}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}.$$

(iii) Let $\Phi(\beta,p)>0$ and $\vartheta(\beta,p)<0$, and then $p_0=\sqrt{\frac{-\vartheta(\beta,p)}{2\,\Phi(\beta,p)}}$ is a critical point. Let $p_0\in(0,2)$. Since $E''(\beta,p_0)>0$, p_0 is local minimum point of $E(\beta,p)$. Therefore, $E(\beta,p)$ cannot have a local maximum.

(iv) If $\Phi(\beta,p)<0$ and $\vartheta(\beta,p)>0$, then p_0 is critical point of $E(\beta,p)$. Since $E''(\beta,p_0)<0$, p_0 is local maximum point for $p_0\in(0,2)$ and so the maximum value occurs at $p=p_0$. Hence,

$$max\{E(\beta,p):p\in(0,2)\}=E(\beta,p_0),$$

where

$$E(\beta,p_0)=\frac{4(1-\beta)^2}{(2m+1)^2[2\tau(1-m)+1]^2\theta_{2m+1,\delta}^2}-\frac{(1-\beta)^2\,\vartheta^2(\beta,p)}{8\,\Phi(\beta,p)}.$$

Remark 3.1. For $\tau=0$, a function $f\in\sum$ belongs to the class $\mathcal{C}_{\Sigma m}\,(\tau;\beta)$ if it fullfils

$$Re\left[\frac{z\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda}\,f(z)\right)'}{z}\right]>\beta,$$

$$Re\left[\frac{w\left(\mathcal{J}_{s,a,\mu}^{\delta,\lambda}\,g(w)\right)'}{w}\right]>\beta.$$

If we let $\tau=0$, we arrive at Corollary 3.1.

Corollary 3.1. Let f given by (1) belong to the class $\mathcal{C}_{\Sigma m}\,(0\,;\beta)=\mathfrak{D}_{\Sigma m}\,(\beta)$. Then

$$|\,a_{m+1}a_{3m+1}-a_{2m+1}^2|\le\begin{cases}E(\beta,2-), & if\ \Phi(\beta,p)\ge 0 \text{ and } \vartheta(\beta,p)\ge 0\\ \dfrac{4(1-\beta)^2}{(2m+1)^2\theta_{2m+1,\delta}^2}, & if\ \Phi(\beta,p)\le 0 \text{ and } \vartheta(\beta,p)\le 0\\ max\left\{\dfrac{4(1-\beta)^2}{(2m+1)^2\theta_{2m+1,\delta}^2},E(\beta,2-)\right\}, & if\ \Phi(\beta,p)>0 \text{ and } \vartheta(\beta,p)<0\\ \max\{E(\beta,p_0),E(\beta,2-)\}, & if\ \Phi(\beta,p)<0 \text{ and } \vartheta(\beta,p)>0\end{cases},$$

where

$$E(\beta,2-)=\frac{4(1-\beta)^2}{(2m+1)^2\,\theta_{2m+1,\delta}^2}+(1-\beta)^2\,[8\Phi(\beta,p)+2\vartheta(\beta,p)],$$

$$E(\beta,p_0)=\frac{4\,(1-\beta)^2}{(2m+1)^2\,\theta_{2m+1,\delta}^2}-\frac{(1-\beta)^2\,\vartheta^2(\beta,p)}{8\,\Phi(\beta,p)},\qquad p_0=\sqrt{\frac{-\vartheta(\beta,p)}{2\,\Phi(\beta,p)}},$$

$$\Phi(\beta,p)=\frac{(1-\beta)^2}{2(m+1)^2\,\theta_{m+1,\delta}^4}-\frac{(3m+2)(1-\beta)}{2(m+1)^2(m+1)}-\frac{1}{(m+1)(3m+1)\,\theta_{m+1,\delta}\,\theta_{3m+1,\delta}}+\frac{1}{2(2m+1)^2\,\theta_{2m+1,\delta}{}^2},$$

$$\vartheta(\beta,p)=\frac{2(3m+2)(1-\beta)}{(m+1)^2(2m+1)}+\frac{6}{(m+1)(3m+1)\theta_{m+1,\delta}\theta_{3m+1,\delta}}-\frac{4}{(2m+1)^2\theta_{2m+1,\delta}^2}.$$

4. CONCLUSION

In the study carried out, a new operator was described via convolution. By using this operator, a new class was presented and studied.

For the future studies, it is planned to describe new subclasses of *m*-fold symmetric bi-univalent functions. It is also planned to present upper bounds for initial Taylor coefficients, Fekete-Szegö and Hankel determinant inequalities for functions in the defined classes.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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